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Algebra 1

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Quiz #3

Problems:

1. (20 pt) Let G be a group acting on a set X.

   (a) If $|G| = 11$ and $|X| = 12$ prove that the action has at least one fixed point.
   Solutions: *Since* $|G| = 11$, *the orbits can have* 1 *or* 11 *elements. Since* $|X| = 12$ *we have*
   $$12 = |X| = \sum_{Pdistinct Ox} |Ox|$$
   So, we have either $12 = 11+1$ *or* $12 = 1+1+1+1+1+1+1+1+1+1+1+1$. *These are the only possibilities and in both cases there is at least one* $x \in X$ *such that the orbit* $|Ox| = 1$ *has only one element, hence* G *fixes that point, i.e.* x *is a fixed point of the action.*

   (b) If $|G| = 11$ and $|X| = 10$ prove that $G(x) = x$ for all $x \in X$.
   Solutions: *Since* $|G| = 11$, *the orbits can have* 1 *or* 11 *elements. Since* $|X| = 10$ *we have*
   $$10 = |X| = \sum_{Pdistinct Ox} |Ox|$$
   So, we have $10 = 1+1+1+1+1+1+1+1+1+1$.
   So for each $x \in X$ *the orbit* $|Ox| = 1$ *has only one element, hence* G *fixes that point, i.e.* $g(x) = x$ *for all* $g \in G$, *or* $G(x) = x$.
   Hence for each $x \in X$ *we have* $G(x) = x$.

2. (30 pt)

   (a) Prove that $\{(124), (142), (1)\}$ is a subgroup of S_4.
   Solutions:
   Let $H = \{(1), (124), (142)\}$. *Then:*

   i. *H is subset of S_4 by definition.*

   ii. *H is nonempty since it has some elements.*

   iii. *H is closed under operation as can be seen on the following table:*

H	(1)	(124)	(142)
(1)	(1)	(124)	(142)
(124)	(124)	(142)	(1)
(142)	(142)	(1)	(124)

iv. *S closed under inverses:* $(1)^{-1} = (1)$, $(124)^{-1} = (142)$, $(142)^{-1} = (124)$

(b) Prove that $\{(124), (142), (1)\}$ is not a normal subgroup of S_4.
Solutions:
Let $K = \{(124), (142), (1)\}$. *It is enough to find at least one* $a \in S_4$ *such that* $aKa^{-1} \subsetneq K$.

$$(13)K(13)^{-1} = \{(13)(124)(13)^{-1}, (13)(142)(13)^{-1}, (13)(1)(13)^{-1}\} = \{(324), (342), (1)\}$$

Therefore $(13)K(13)^{-1} \subsetneq K$. *So* K *is not normal subgroup of* S_4.

(c) Describe all elements of order 4 in S_4.
Solutions:
There are all cycles of length 4. *There are* $\binom{4}{4}\frac{4!}{4} = 6$ *such permutations:*

$$\{(1,2,3,4), (1,2,4,3), (1,3,2,4), (1,3,4,2), (1,4,2,3), (1,4,3,2)\}$$

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